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Boundaries of cocompact proper CAT(0) spaces

Ross Geoghegan*, Pedro Ontaneda

Binghamton University (SUNY), Binghamton, NY 13902, USA

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Abstract

A proper CAT(0) metric space X is *cocompact* if it has a compact generating domain with respect to its full isometry group. Any proper CAT(0) space, cocompact or not, has a compact metrizable boundary at infinity $\partial_\infty X$; indeed, up to homeomorphism, this boundary is arbitrary. However, cocompactness imposes restrictions on what the boundary can be. Swenson showed that the boundary of a cocompact X has to be finite-dimensional. Here we show more: the dimension of $\partial_\infty X$ has to be equal to the global Čech cohomological dimension of $\partial_\infty X$. For example: a compact manifold with non-empty boundary cannot be $\partial_\infty X$ with X cocompact. We include two consequences of this topological/geometric fact: (1) The dimension of the boundary is a quasi-isometry invariant of CAT(0) groups. (2) Geodesic segments in a cocompact X can "almost" be extended to geodesic rays, i.e. X is almost geodesically complete.

1. Introduction

A CAT(0) space is a geodesic metric space (X, d_X) whose geodesic triangles are "no fatter than" the corresponding comparison triangles in the Euclidean plane. A general reference for facts about CAT(0) spaces used here is [5]. We will usually suppress d_X referring just to X. Such a space X is *proper* if all closed balls are compact, and is *cocompact* if there is a compact generating domain for the full isometry group of X, i.e. there is a compact set $C \subset X$ such that the sets $\{h(C) \mid h \text{ is an isometry of } X\}$ cover X. In particular, a proper CAT(0) space X has a compact *boundary*, $\partial_\infty X$, namely the set of asymptoty

* Corresponding address: Binghamton University, Department of Mathematical Sciences, 13902-6000 NY, USA. Tel.: +1 607 798 0787; fax: +1 607 777 2450.

E-mail addresses: ross@math.binghamton.edu (R. Geoghegan), pedro@math.binghamton.edu (P. Ontaneda).

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classes of geodesic rays in X with the "cone topology". Equivalently, picking a base point $p \in X$ one can regard $\partial_\infty X$ as the set of geodesic rays starting at p endowed with the compact-open topology.

There is no restriction for a compact metrizable space to be homeomorphic to the boundary of a proper CAT(0) space; that is, every compact metrizable space is homeomorphic to the boundary of a proper CAT(0) space (see the Appendix). But one can ask: can any compact metrizable space be homeomorphic to the boundary of a proper *cocompact* CAT(0) space? The answer is no. First, there is the following theorem of E.L. Swenson:

Theorem 0 (*[17]; Theorem 12*). *If a space Y is homeomorphic to the boundary of a proper cocompact* CAT(0) *space then the dimension of Y is finite.*

Here, "dimension" means Lebesgue covering dimension (see [14]); we recall the precise definition in Section 2.

In this paper we show that cocompactness forces another important restriction:

Main Theorem. *Let the non-empty compact metrizable space Y be homeomorphic to the boundary of a proper cocompact* CAT(0) *space and let its dimension be d (necessarily finite in view of Theorem 0). Then the d-dimensional reduced Čech cohomology of Y with integer coefficients is non-trivial.*

This theorem and its proof are in the spirit of [4] where an analogous statement is proved for the boundaries of hyperbolic groups.

Remark 1. Since the Čech cohomology of a d-dimensional compact metrizable space vanishes in dimensions above d, one could reword this as saying: the "global cohomological dimension" of the boundary and the (locally defined) dimension of the boundary are finite and coincide.

Remark 2. Let $\bar{X} = X \cup \partial_\infty X$, with the usual compact metrizable topology — see page 263 of [5]. Then $\bar{X}$ is a Z-set compactification of X. (Recall that this means: For every open set U in $\bar{X}$ the inclusion map $U - \partial_\infty X \to U$ is a homotopy equivalence.) To see that $\partial_\infty X$ is indeed a Z-set, observe that $\partial_\infty X$ can be regarded as the space of geodesic rays in X starting at some base point $p \in X$, and each point $q \in X$ can be identified with the "generalized geodesic ray" which proceeds geodesically from p to q and then stays at q for the rest of time; the Z-set property is then obvious — just retract geodesic rays to generalized geodesic rays as needed. It follows from the Z-set property that the Čech cohomology of the compact space $\partial_\infty X$ is canonically isomorphic to the direct limit of the cohomology groups $\{H^*(X - K)\}$ as K varies over the (directed set of) compact subsets of X. This in turn implies, by straightforward homology arguments given, for example, in [9,10,4,3,11], that $H^n_c(X)$ (integral cohomology with compact supports) is isomorphic to the reduced integral $(n-1)$-dimensional Čech cohomology of $\partial_\infty X$. Thus, one could reword the Main Theorem as saying that "global cohomological dimension with respect to compact supports" of a proper cocompact CAT(0) space X is equal to $1 + \dim(\partial_\infty X)$.

Example A. If M^n is a compact n-manifold with non-empty boundary, then its dimension is n, but its n-dimensional (Čech) cohomology is trivial. Hence, by the Main Theorem, M^n cannot be homeomorphic to the boundary of a proper cocompact CAT(0) space. For example: $\mathbb{S}^n \times [0,1]$ cannot be homeomorphic to the boundary of a proper cocompact CAT(0) space, even though $\mathbb{S}^n$ is, for example, homeomorphic to the boundary of any $(n+1)$-dimensional Hadamard manifold.

Example B. Let Q denote the Hilbert cube with metric inherited from Hilbert space. The proper CAT(0) spaces $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$ and "the open cone on Q" (made into a CAT(0) space in the way described in [5] II.3.14) have contractible boundaries homeomorphic to B^n and Q respectively, so the Main Theorem implies that they are not cocompact. This is because the reduced Čech cohomology of the boundary is trivial, while the dimension of the boundary is non-negative. (Recall that only the empty space has negative dimension, namely -1, and the boundary of a proper CAT(0) space X is empty if and only if X is compact.)

Remark 3. A proper cocompact CAT(0) space can have infinite dimension. For instance $\mathbb{R} \times Q$ is such a case; but of course its boundary is homeomorphic to $\mathbb{S}^0$.

Here is an application in geometric group theory. A *CAT(0) group* is a group Γ which can act *geometrically* (i.e. properly discontinuously and cocompactly by isometries) on some proper CAT(0) space X. Then $H_c^*(X)$ is isomorphic to $H^*(\Gamma, \mathbb{Z}\Gamma)$ (see Exercise VIII.7.4 of [6]). Thus, by the (remarks following the) Main Theorem, if the dimension of $\partial_\infty X$ is d then $H^{d+1}(\Gamma, \mathbb{Z}\Gamma)$ is non-zero while $H^n(\Gamma, \mathbb{Z}\Gamma)$ is trivial whenever $n > d + 1$. Now, if Γ_1 and Γ_2 are quasi-isometric CAT(0) groups then $H^*(\Gamma_1, \mathbb{Z}\Gamma_1)$ and $H^*(\Gamma_2, \mathbb{Z}\Gamma_2)$ are isomorphic [12]. Thus the dimension of $\partial_\infty X$, depends only on the quasi-isometry type of Γ. This is of interest because Croke and Kleiner [7] have given examples to show that the homeomorphism type of $\partial_\infty X$ is not an invariant of Γ. Summarizing:

Corollary 1. *The dimension of the boundary of a proper* CAT(0) *space on which a* CAT(0) *group Γ acts geometrically is a quasi-isometry invariant of Γ.*

Remark 4. Of course, if Γ acts cocompactly as covering transformations on X then Γ has finite dimension (in the group theoretic sense of [6]) so the number d of the Main Theorem exists for group theoretic reasons, and in that case Corollary 1 follows from the proof of Corollary 1.4 of [4], as is pointed out in [3].

The next corollary follows from Theorem 0, the Main Theorem and Remark 2.

Corollary 2. *Let X be a non-empty proper cocompact* CAT(0) *space. Then the cohomology with compact supports of X is non-trivial.*

Corollary 2 has an interesting geometric consequence. We say the CAT(0) space X is *almost geodesically complete* if there is a number $r \geq 0$ such that for any points a and b of X there is a geodesic ray $\gamma : [0, \infty) \to X$ with $\gamma(a) = 0$ whose image meets the open ball about b of radius r. The term "almost extendible" is also used. Obviously, the spaces in Example A have this property while those in Example B do not. A cocompact example which has this property even though not every geodesic segment can be extended to a geodesic ray is the graph consisting of $\mathbb{R}$ together with, for each $n \in \mathbb{Z}$, a copy of the closed unit interval glued at its 0-point to $n \in \mathbb{R}$.

Corollary 3. *Every non-compact cocompact proper* CAT(0) *space is almost geodesically complete.*

Remark 5. It is shown in [15] that Corollary 3 follows directly from Corollary 2. The version of Corollary 2 proved in [15] has the additional hypothesis that for some cocompact group of isometries Γ the orbits of the Γ-action are discrete; see footnote on page 209 of [5]. Our Corollary 2 removes the discreteness hypothesis. Still the version of Corollary 2 proved in [15] is much stronger in the sense that no geometric assumptions are required (like being CAT(0) or having a "space at infinity"; see Proposition B of [15]).

Remark 6. Theorem H of [2] requires almost geodesic completeness as a hypothesis on the non-compact cocompact proper CAT(0) spaces under consideration there. Corollary 2 shows that this hypothesis is redundant.

Recently, Dranishnikov has established in [8] the conclusion of our Main Theorem in the following context: a discrete group Γ acts properly and cocompactly on a metric space X having a "Higson dominated" compactification $\bar{X} = X \cup C$, where $\bar{X}$ is an absolute retract and C is a Z-set. When Γ is torsion free this reduces to a case covered in [3].

2. Proof of the Main Theorem

We denote the Alexander–Spanier cohomology of the pair (Z, A) by $\check{H}^n(Z, A)$.

Lemma 1. *Let Z be a compact metric space and A a closed subset of Z. Assume that $\check{H}^n(Z, A) \neq 0$ and that $\check{H}^{n+1}(Z, B) = 0$ for every closed subset B of Z. Then there is a sequence of open balls $\{B_k\}_{k \geq k_0}$ in $Z \setminus A$, B_k of radius $1/k$, such that the homomorphisms $\check{H}^n(Z, Z \setminus B_k) \to \check{H}^n(Z, A)$ are non-zero. Moreover, we can choose the B_k's to satisfy $d_Z(B_k, A) \geq \delta$, for some $\delta > 0$.*

Proof. Since $\check{H}^*(Z, A) = \check{H}^*_c(Z \setminus A)$ (see 6.6.11 of [16]), elements can be represented by cocycles with compact support in $Z \setminus A$. Hence we can find a closed set A', such that $A \subset \operatorname{int} A'$ and the morphism $\check{H}^n(Z, A') \to \check{H}^n(Z, A)$ is non-zero. Let k_0 be such that $2/k_0 < d_Z(Z \setminus A', A)$.

Let $U_1, \ldots, U_j$ be a finite cover of $\overline{Z \setminus A'}$ by balls of radius $1/k_0$. Write $U = U_1$ and $V = \bigcup_{i=2}^{j} U_i$. Note that $A \subset Z \setminus (U \cup V) \subset A'$. We have the following diagram of Mayer–Vietoris sequences:

$$\begin{array}{ccccccc}
\check{H}^n(Z, Z \setminus U) & \oplus & \check{H}^n(Z, Z \setminus V) & \to & \check{H}^n(Z, Z \setminus (U \cup V)) & \to & \check{H}^{n+1}(Z, Z \setminus (U \cap V)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\check{H}^n(Z, A) & \oplus & \check{H}^n(Z, A) & \to & \check{H}^n(Z, A) & \to & \check{H}^{n+1}(Z, A)
\end{array}$$

where the last group of the first row is zero, by hypothesis. Since the non-zero morphism $\check{H}^n(Z, A') \to \check{H}^n(Z, A)$ factors through $\check{H}^n(Z, Z \setminus (U \cup V))$, the middle vertical morphism in the diagram is non-zero. It follows from the diagram that (at least) one of the morphisms $\check{H}^n(Z, Z \setminus U) \to \check{H}^n(Z, A)$ and $\check{H}^n(Z, Z \setminus V) \to \check{H}^n(Z, A)$ is non-zero. If the first one is non-zero take $B_{k_0} = U$. Otherwise write $U = U_2$ and $V = \bigcup_{i=3}^{j} U_i$, and repeat the process (using the fact that the latter morphism is non-zero). Eventually, we will find an i such that the morphism $\check{H}^n(Z, Z \setminus U_i) \to \check{H}^n(Z, A)$ is non-zero. Take $B_{k_0} = U_i$. To find B_{k_0+1} take a finite cover $U_1, \ldots, U_j$ of $\bar{B}_{k_0}$ by balls of radius $\frac{1}{k_0+1}$ and repeat the process. In this way we obtain a sequence of balls B_k. Note that $d_Z(B_k, A) \geq d_Z(A, \overline{Z \setminus A'}) - 1/k_0 > 0$. Take $\delta = d_Z(A, \overline{Z \setminus A'}) - 1/k_0$. $\square$

Two metric spaces *have the same bounded homotopy type* if there are maps (i.e. continuous functions) from each to the other such that either composition is homotopic to the appropriate identity map by a homotopy which only moves points by a bounded amount, i.e. there is a number $s \geq 0$ such that the image of *point* $\times$ $[0, 1]$ under either of the homotopies has diameter at most s. When that happens the maps in question are called *bounded homotopy equivalences*. If the metric spaces are proper (i.e. all closed balls are compact) then these must be proper; moreover, "having the same bounded homotopy type" implies "having the same proper homotopy type" and a bounded homotopy equivalence is a proper homotopy equivalence. In particular, H_c^* is a bounded homotopy invariant.

When K is a countable locally finite simplicial complex, $|K|$ will be understood to carry the *unit metric* unless we say otherwise: by this we mean that each simplex is isometric to a standard Euclidean simplex whose edges have length 1, and the distance between points of $|K|$ is the greatest lower bound of the lengths of all piecewise linear paths joining them such that each linear piece lies in a simplex. Thus $|K|$ is a proper geodesic metric space.

Theorem 2. *Let X be a cocompact proper* CAT(0) *space. There is a finite-dimensional countable locally finite simplicial complex K such that X and $|K|$ have the same bounded homotopy type.*

A version of Theorem 2 appears in [15] but with the additional hypothesis that for some cocompact group Γ of isometries of X the orbits of the Γ-action are discrete; see footnote on page 209 of [5]. Here we remove that discreteness hypothesis. (Of course the version in [15] gives a Γ-complex K, while we are not claiming that the complex K in Theorem 2 supports a group action.)

Proof of Theorem 2. Let $E \subset X$ be maximal with respect to the property that if $x, y \in E$ and $x \neq y$, then $d_X(x, y) \geq 1$. The family $\mathcal{U} = \{B_X(x, 1) \mid x \in E\}$ is an open cover of X, where $B_X(x, 1)$ denotes the open ball of radius 1. Let K be the nerve of this cover. Then K is clearly countable and it is locally finite because the cover $\mathcal{U}$ is star finite (i.e. each member of $\mathcal{U}$ meets only finitely many others).

Suppose K is not finite-dimensional. Then for all natural numbers m we would have $\dim K \geq m$. Thus for each m there would be points $x_0^m, \ldots, x_m^m \in E$ such that $1 \leq d_X(x_i^m, x_j^m) < 2$ when $i \neq j$. Let C be a compact generating domain for X. For each m there is an isometry h_m such that $h_m(x_0^m) \in C$. Then for each i the point $h_m(x_i^m)$ lies in C_1, the 1-neighborhood of C. By induction, we can then pick subsequences of $\mathbb{N}$, each a subsequence of its predecessor, so that for each n the sequence $(h_{n+k}(x_n^{n+k}))$ converges to some point $y_n \in C_1$. The resulting sequence (y_n) consists of points which are pairwise at least 1 apart while all lying in the compact set C_1, a contradiction.

Because X is CAT(0) and $\mathcal{U}$ consists of open balls, it follows that all non-empty intersections of members of $\mathcal{U}$ are contractible. It is well known that this implies that X is homotopy equivalent to $|K|$ (the weak topology coincides with the unit metric topology since K is locally finite), and indeed the proof of this fact in Section 5 of [18] shows that, with the metric we have chosen, the mutually homotopy inverse maps in both directions given in that proof are bounded homotopy equivalences which are bounded homotopy inverse to one another. (Indeed, this also follows from the proof of Lemma 7A.15 on page 129 of [5]). $\square$

When Z is a metric space $B_Z(p, r)$ denotes the open ball of radius r and center $p \in Z$. We say that Z is *uniformly contractible* if for every $r > 0$ there is $s > 0$ such that for every $p \in Z$, $B_Z(p, r)$ contracts in $B_Z(p, s)$. We say $H_c^i(Z)$ *is uniformly trivial* if Z has the following property:

For every $r > 0$ there is $s > 0$ such that whenever an i-cocycle z has compact support contained in a ball $B_Z(p, r)$, then z cobounds a cochain whose compact support lies in $B_Z(p, s)$.

Recall that when $U \subset V \subset Z$ with U and V both open in Z then the inclusion map $\iota : U \to V$ induces a homomorphism $\iota_* : H_c^*(U) \to H_c^*(V)$; see Remark 26.2 of [13]. Hence, $H_c^*(Z)$ is uniformly trivial if for every $r > 0$ there is $s > 0$ such that the map $H_c^*(B_Z(p, r)) \to H_c^*(B_Z(p, s))$, induced by the inclusion, is trivial.

Proposition 1. *Let X be a proper cocompact* CAT(0) *space. If $H_c^i(X)$ is trivial, then $H_c^i(X)$ is uniformly trivial. In fact the number s in the definition of "uniformly trivial" is independent of i.*

Proof. Let $r > 0$ and let $x \in X$.

Claim. There is $r' \geq r$ (depending only on r) such that $\iota_*(H_c^i(B_X(x,r)))$ is a finitely generated subgroup of $H_c^i(B_X(x,r'))$, where $\iota : B_X(x,r) \to B_X(x,r')$ is the inclusion.

To prove the claim, let K be as in the proof of Theorem 2 and let $f : X \to |K|$, $g : |K| \to X$ be such that gf is bounded homotopic to the identity map. Then there is a compact set C containing $B_X(x,r)$ such that, for any cocycle z with compact support lying in $B_X(x,r)$, z and f^*g^*z are compactly cohomologous in C.

Let L be a finite subcomplex of K such that:

$$g^{-1}(C) \subset \operatorname{int}|L|. \tag{$*$}$$

Choose r' such that $g(|L|) \subset B_X(x,r')$. It follows that $C \subset g(\operatorname{int}|L|) \subset B_X(x,r')$. Hence, if z is a cocycle with compact support contained in $B_X(x,r)$, then z is compactly cohomologous to $(gf)^*z = f^*g^*z$ in $B_X(x,r')$. By $(*)$ $v := g^*z$ is a cocycle with compact support contained in $\operatorname{int}|L|$. Consequently every cocycle with compact support contained in $B_X(x,r)$ is compactly cohomologous in $B_X(x,r')$ to a cocycle of the form f^*v where v is a cocycle with compact support contained in $\operatorname{int}|L|$. Since $H_c^i(\operatorname{int}|L|)$ is finitely generated we conclude that $f^*H_c^i(\operatorname{int}|L|)$ is also finitely generated. The claim follows.

Now let r' be as in the claim and let $z_1, \ldots, z_l$ be compactly supported cocycles representing a set of generators of $\iota_*(H_c^i(B_X(x,r)))$. Since we are assuming $H_c^i(X) = 0$, there is $s = s(r)$ such that $z_1, \ldots, z_l$ compactly cobound in $B_X(x,s)$. Since X is cocompact it follows easily that, given r, a number s independent of x exists with the required property. That s is independent of i follows from the fact that K in Theorem 2 is finite-dimensional. $\square$

The following is a consequence of Proposition 1 and the construction of the chain homotopy Δ^i given in Proposition 1.4(a) of [4].

Corollary. *Let X be a proper cocompact* CAT(0) *space. Assume that there is a k such that $H_c^i(X) = 0$ for $i > k$. Then there is a number t such that every i-cocycle with compact support cobounds in the t-neighborhood of its support, $i > k$.*

Remark. The construction of Δ^i given in [4] is done for a finite-dimensional simplicial complex, but this is not a problem since, by Theorem 2, X is bounded homotopy equivalent to a finite-dimensional simplicial complex.

We now recall some facts of dimension theory; see [14] for details. Let Z be a compact metrizable space. The (Lebesgue covering) *dimension* of Z, $\dim Z$, is $\leq n$ if every open cover of Z has a refinement whose nerve has dimension at most n; one writes $\dim Z = n$ if, in addition, $\dim Z$ is not $\leq n-1$. If there is no such n then $\dim Z$ is infinite. The $\mathbb{Z}$-*cohomological dimension* of Z, $\dim_{\mathbb{Z}} Z$, is n if $\check{H}^n(Z,A) \neq 0$ for some closed subset A of Z while for all $k > n$ and all closed subsets B of Z, $\check{H}^k(Z,B) = 0$. If there is no such n then $\dim_{\mathbb{Z}} Z$ is infinite. Traditionally, here, $\check{H}^n$ refers to Čech cohomology, but that is canonically isomorphic to Alexander–Spanier cohomology, at least when $\dim Z$ is finite, as it is in our case — see page 342 of [16]. There is always the inequality $\dim_{\mathbb{Z}} Z \leq \dim Z$, and equality holds when $\dim Z$ is finite.

Proof of the Main Theorem. We write $\bar{X} = X \cup \partial_\infty X$. Let d be the largest integer such that $H_c^{d+1}(X) \neq 0$ (if $H_c^*(X) = 0$, then $d = -\infty$). By Theorem 0 we know that $\dim \partial_\infty X$ is finite so we may use the cohomological definition of dimension. Write $n = \dim_{\mathbb{Z}} \partial_\infty X \geq 0$. Then $n \geq d$. We have to show that $n = d$, so we will suppose that $n > d$ and derive a contradiction. There must be a closed set A such that $\check{H}^n(\partial_\infty X, A) \neq 0$. Let B_k and $\delta > 0$ be as in Lemma 1 (taking $Z = \partial_\infty X$ with a metric $d_{\partial_\infty X}$ that induces the cone topology on $\partial_\infty X$). We may assume that the balls B_k converge to some point $\gamma \notin A$.

Fix a base point $x_0 \in X$. For any set $G \subset \partial_\infty X$, the cone CG of G is the union of all rays emanating from x_0 and ending in G. We claim that, using geodesic retraction along rays emanating from x_0, we can find a closed set $D \subset \bar{X}$ such that:

(i) $D \cap \partial_\infty X = A$ and $CA \subset D$.
(ii) $D \setminus A$ lies in the 1-neighborhood (in X) of $CA \setminus A$.
(iii) D is a strong deformation retract of $\bar{X}$.

Here are the details of how to construct this set D. We will abbreviate the geodesic ray $[x_0, \beta]$ to β. Then $\beta(0) = x_0$, $\beta(\infty) = \beta$ and $d_X(\beta(t), x_0) = t$. Let $\tau(\beta) = \inf\{t : d_X(\beta(t), CA) \geq 1\}$. Note that the function $\tau : \partial_\infty X \to [0, \infty]$ is not necessarily continuous. This function τ has the following properties:

(1) $\tau(\beta) < \infty$ when $\beta \notin A$, and $\tau(\beta) = \infty$ when $\beta \in A$.
(2) If $\beta_n \in \partial_\infty X$ is such that $d_{\partial_\infty X}(\beta_n, A) \to 0$ then $\tau(\beta_n) \to \infty$.
(3) $d_X(\beta(\tau(\beta)), CA) = 1$ when $\beta \notin A$. Moreover, $d_X(\beta(t), CA) \leq 1$ when $t \leq \tau(\beta)$ and $\beta \notin A$.

For every $n = 1, 2, 3 \ldots$ let $c_n = \inf\{\tau(\beta) : \frac{1}{n+1} \leq d_{\partial_\infty X}(\beta, A) \leq \frac{1}{n}\}$. By (1) $c_n < \infty$, and by (2) $c_n \to \infty$, as $n \to \infty$. Let $\vartheta : (0, \infty) \to [0, \infty)$ be a continuous function such that:

(a) $\vartheta(s) = 0$, for $s \geq 1$.
(b) $\vartheta(s) \leq c_n$, for $s \in [\frac{1}{n+1}, \frac{1}{n}]$.
(c) $\vartheta(s) \to \infty$ as $s \to 0^+$.

Then we can extend ϑ by $\vartheta(0) = \infty$. Let $D = \{\beta(t) : \beta \in \partial_\infty X,\ t \leq \vartheta(d_{\partial_\infty X}(\beta, A))\}$. Since $\vartheta(\beta) \leq \tau(\beta)$, (1) implies (i). Also, (b) and (3) imply (ii). Finally, we can deform $\bar{X}$ to D by deforming each ray β to $\beta([0, \vartheta(d_{\partial_\infty X}(\beta, A))])$. Thus D satisfies (i), (ii) and (iii) as claimed.

Let t be as in the corollary to Proposition 1. Since $\gamma \notin A$ we can find a point x_1 in $[x_0, \gamma]$, such that the ball $B_X(x_1, t+2)$ does not intersect D. Write $a = d_X(x_0, x_1)$. Since $B_k \to \gamma$ there is a k' such that every geodesic ray in $C\bar{B}_{k'}$ intersects $B_X(x_1, 1) \cap S_X(x_0, a)$, where $S_X(x_0, a)$ denotes the sphere of radius a centered at x_0. We write $B' = B_{k'}$. By Lemma 1, $\check{H}^n(\partial_\infty X, \partial_\infty X \setminus B') \to \check{H}^n(\partial_\infty X, A)$ is non-zero. Let $\{z\} \in \check{H}^n(\partial_\infty X, \partial_\infty X \setminus B')$ be such that its image in $\check{H}^n(\partial_\infty X, A)$ is non-zero, where z denotes a cocycle with compact support lying in B'. (Here, as at the beginning of the proof of Lemma 1, we are identifying a relative Čech cohomology group with the compactly supported cohomology of the complement — we will use this convention again below.) Using a geodesic deformation retraction we can find a closed set $E \subset \bar{X}$ such that:

(i) $E \cap \partial_\infty X = \partial_\infty X \setminus B'$.
(ii) $C(\partial_\infty X \setminus B') \cup B_X(x_0, a) \subset E$.

The construction of E is similar to the construction of D given above.

Note that the distance from $D \setminus A$ to $X \setminus E$ is larger than $t + 1$. From the exact sequence of the triple $(\bar{X}, \partial_\infty X \cup D, D)$ and the fact that $\check{H}^*(\bar{X}, D) = 0$ we conclude that $\check{H}^n(\partial_\infty X \cup D, D) \to \check{H}^{n+1}(\bar{X}, \partial_\infty X \cup D)$ is an isomorphism. By considering also the exact sequence of the triple $(\bar{X}, \partial_\infty X \cup E, E)$ and the inclusion $(\bar{X}, \partial_\infty X \cup D, D) \hookrightarrow (\bar{X}, \partial_\infty X \cup E, E)$ we get the following commutative diagram:

$$\begin{array}{ccccc}
\check{H}^n(\partial_\infty X, \partial_\infty X \setminus B') & \cong & \check{H}^n(\partial_\infty X \cup E, E) & \to & \check{H}^{n+1}(\bar{X}, \partial_\infty X \cup E) \\
\downarrow & & \downarrow & & \downarrow \\
\check{H}^n(\partial_\infty X, A) & \cong & \check{H}^n(\partial_\infty X \cup D, D) & \cong & \check{H}^{n+1}(\bar{X}, \partial_\infty X \cup D)
\end{array}$$

where the isomorphisms on the left are given by excision — see 6.6.5 of [16]. Let $\{z'\}$ be the image of $\{z\}$ in $\check{H}^{n+1}(\bar{X}, \partial_\infty X \cup E)$. Then the image of $\{z'\}$ in $\check{H}^{n+1}(\bar{X}, \partial_\infty X \cup D) \cong \check{H}_c^{n+1}(X \setminus D)$ is non-zero. We regard z' as a cocycle compactly supported outside $\partial_\infty X \cup E$ and we obtain a contradiction by showing that z' compactly cobounds outside $D \setminus A$ as follows: since the support of z' lies in $X \setminus E$ and the distance from $D \setminus A$ to $X \setminus E$ is larger than $t + 1$, the corollary to Proposition 1 implies that z' cobounds outside $D \setminus A$, a contradiction. □

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Appendix

That cocompactness is essential in the theorems proved here is shown by the following observation attributed to Gromov — see the remark preceding 2.28 of [1]:

Proposition 2. *Let C be a compact metrizable space. There is a proper* CAT(0) *space, indeed, a proper* CAT(−1) *space, X, such that $\partial_\infty X$ is homeomorphic to C.*

Proof (*Sketch*). For the case where C is finite-dimensional, embed it in a suitably high-dimensional sphere which is regarded as the boundary of a hyperbolic space H. Let X be the closed convex hull of the union of all geodesic lines in H joining two points of C. Then X has the required properties. For the infinite-dimensional case, do the same using an infinite-dimensional hyperbolic space.

References

[1] N. Benakli, I. Kapovich, Boundaries of hyperbolic groups, in: R. Gilman, et al. (Eds.), Combinatorial and Geometric Group Theory, in: Contemporary Mathematics, vol. 296, 2002, pp. 39–94.

[2] R. Bieri, R. Geoghegan, Connectivity properties of group actions on non-positively curved spaces, Mem. Amer. Math. Soc. 161 (765) (2003).

[3] M. Bestvina, Local homology properties of boundaries of groups, Michigan Math. J. 43 (1) (1996) 123–139.

[4] M. Bestvina, G. Mess, The boundary of negatively curved groups, J. Amer. Math. Soc. 4 (3) (1991) 469–481.

[5] M. Bridson, A. Haefliger, Metric Spaces of Non-Positive Curvature, in: Grundlehren der Mathematischen Wissenschaften, vol. 319, Springer-Verlag, Berlin, 1999.

[6] K.S. Brown, Cohomology of Groups, in: Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, Berlin, 1982.

[7] C. Croke, B. Kleiner, Spaces with nonpositive curvature and their ideal boundaries, Topology 39 (3) (2000) 549–556.

[8] A. Dranishnikov, On Bestvina–Mess formula, in: Topological and Asymptotic Aspects of Group Theory, in: Contemporary Mathematics, vol. 394, American Mathematical Society, Providence, 2006.

[9] R. Geoghegan, Topological methods in group theory, monograph (in preparation).

[10] R. Geoghegan, The shape of a group-connections between shape theory and the homology of groups, in: Geometric and Algebraic Topology, in: Banach Center Publ., vol. 18, PWN, Warsaw, 1986, pp. 271–280.

[11] R. Geoghegan, M. Mihalik, Free abelian cohomology of groups and ends of universal covers, J. Pure Appl. Algebra 36 (1985) 123–137.
[12] S. Gersten, Quasi-isometry invariance of cohomological dimension, C. R. Acad. Sci. Paris Sci., Paris S r. I Math. 316 (5) (1993) 411–416.
[13] M.J. Greenberg, J.R. Harper, Algebraic Topology: A First Course, in: Math. Lecture Notes Series, vol. 58, Benjamin/Cummings Publishing Co., Inc., 1981.
[14] W. Hurewicz, H. Wallman, Dimension Theory, in: Princeton Mathematical Series, vol. 4, Princeton University Press, Princeton, NJ, 1941.
[15] P. Ontaneda, Cocompact CAT(0) spaces are almost geodesically complete, Topology 44 (2005) 47–62.
[16] E.H. Spanier, Algebraic Topology, McGraw-Hill Book Co., New York, Toronto, London, 1966.
[17] E.L. Swenson, A cut point theorem for CAT(0) groups, J. Differential Geom. 53 (1999) 327–358.
[18] A. Weil, Sur les théorèmes de de Rham, Comment. Math. Helv. 26 (1952) 119–145.